

# Analytical solution of an afocal two freeform mirror design problem

PABLO BENITEZ, MILENA NIKOLIC, AND JUAN C. MIÑANO

**Abstract:** We investigate a new afocal two freeform mirror design problem in first order optics. The resulting first-order partial differential equations for the freeform two mirror system have an analytic solution with the sole condition that the  $x$ - $y$  and  $x'$ - $y'$  axes are parallel. Two selected solutions are presented. One of them is semiaplanatic (fulfilling the aplanatic condition only for the  $x$ -coordinates), while the other is, to our knowledge, the first example of an aplanatic two-mirror system without rotational symmetry.

**OCIS codes:** (080.2720) Mathematical methods (general); (080.2740) Geometric optical design; (080.3620) Lens system design; (080.4225) Nonspherical lens design; (080.4298) Nonimaging optics.

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## 1. Introduction

Optical systems that are free from spherical aberration and circular coma are known as aplanatic systems [1]. Aplanatic here refers to full aplanatic, i.e. those aberrations vanish for all orders, and therefore they satisfy the Abbe sine condition. The first rotationally symmetric two mirror aplanatic systems were described by Schwarzschild in 1905 [2] and were later developed and analytically deduced for telescopes which required superior aberration-free imaging [3–6]. One particular aplanatic design is the afocal case, in which the null spherical aberration implies that the plane input wavefront  $WF_{in}$  is coupled to a plane output wavefront  $WF_{out}$  stigmatically. The Abbe sine condition in its general form [7] states that for this afocal case the mapping of the rays between the two wavefronts is not arbitrary, rather that the transformation between Cartesian coordinates  $x$ - $y$  defined on  $WF_{in}$  and Cartesian coordinates  $x'$ - $y'$  defined on  $WF_{out}$ , given by:

$$\begin{aligned}x' &= a_{00}(x, y) \\ y' &= b_{00}(x, y)\end{aligned}$$

must be linear (the sub-indices are used for consistency with equations below).

We will use here the term semi-aplanatic for optical systems in which only one of the functions in Eq. (1) is linear. Since the coordinate systems on the wavefronts can be rotated, without loss of generality it can be considered that the semi-aplanatic case corresponds to the case in which  $a_{00}$  is a linear function of  $x$  only, that is,  $x' = m_x x + l_x$  (where  $m_x$  and  $l_x$  are real numbers), which can be viewed as a one-dimensional Abbe sine condition.

In this paper we will first address a more general afocal design problem in first order optics, in which the ray-to-ray mapping between  $WF_{in}$  and  $WF_{out}$  is such that  $a_{00}$  is a function only of  $x$ , and not necessarily linear. Section 2 is devoted to deriving the first-order angular magnification of the afocal system for this case, using the étendue conservation theorem of bi-parametric ray bundles in phase space. In Section 3 the design problem of two afocal freeform mirrors fulfilling such a ray-to-ray mapping is stated and the resulting first-order partial differential equations are solved in a closed-form. Two specific design examples are given in Section 4, one of them being, to our knowledge, the first example of an aplanatic freeform (i.e. non-rotationally symmetric) two-mirror system. The apparent contradiction with the general rule that three freeform surfaces are needed [8-9] is discussed in Section 5.

## 2. Phase space considerations

Let us consider the plane wavefronts,  $WF_{in}$  and  $WF_{out}$ , which are in general non-parallel, and two Cartesian coordinate systems,  $x$ - $y$ - $z$  and  $x'$ - $y'$ - $z'$ , such that  $WF_{in}$  and  $WF_{out}$  coincide with the planes  $z = 0$  and  $z' = 0$ , respectively and propagate towards positive  $z$  and  $z'$  values.

Let us consider the 4-dimensional phase space at the  $z = 0$  and  $z' = 0$  planes as reference surfaces. Let  $p$  and  $q$  be the optical direction cosines of a ray when crossing the plane  $z = 0$ , and  $p'$  and  $q'$  those at the plane  $z' = 0$ . The afocal optical system will transform a ray  $(x, y, p, q)$  crossing the plane  $z = 0$  onto the ray  $(x', y', p', q')$  crossing  $z' = 0$ , and such a transformation can be expressed in power series of  $p$  and  $q$  as:

$$\begin{aligned}x' &= a(x, y, p, q) = a_{00}(x, y) + a_{10}(x, y)p + a_{01}(x, y)q + \dots \\ y' &= b(x, y, p, q) = b_{00}(x, y) + b_{10}(x, y)p + b_{01}(x, y)q + \dots \\ p' &= c(x, y, p, q) = c_{00}(x, y) + c_{10}(x, y)p + c_{01}(x, y)q + \dots \\ q' &= d(x, y, p, q) = d_{00}(x, y) + d_{10}(x, y)p + d_{01}(x, y)q + \dots\end{aligned}\tag{2}$$

The stigmatic condition (i.e., the rays of  $WF_{in}$  are coupled with those of  $WF_{out}$ ) can be written as  $p' = 0$  and  $q' = 0$  when  $p = 0$  and  $q = 0$ , which means that  $c_{00} = d_{00} = 0$  in Eq. (2).

Applying étendue conservation for bi-parametric bundles as done in [8], we can deduce that the coefficients  $a_{00}$ ,  $b_{00}$ ,  $c_{10}$ ,  $c_{01}$ ,  $d_{10}$  and  $d_{01}$  fulfill:

$$\begin{pmatrix} a_{00x} & b_{00x} \\ a_{00y} & b_{00y} \end{pmatrix} \begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\tag{3}$$

where the subindices  $x$  and  $y$  denote partial derivation. Both matrices on the left hand side of Eq. (3) must therefore be non-singular, in particular:

$$\begin{vmatrix} a_{00x} & b_{00x} \\ a_{00y} & b_{00y} \end{vmatrix} \neq 0 \Rightarrow a_{00x}b_{00y} - a_{00y}b_{00x} \neq 0\tag{4}$$

We are interested in analyzing the implications of imposing the condition that  $a_{00}$  is a function only of  $x$ , i.e.  $a_{00}(x, y) = a_{00}(x)$ . Thus,  $a_{00y} = 0$  and from Eq. (4) it follows that  $b_{00y} \neq 0$ . Hence, expressing Eq. (3) as a set of 4 separate equations, we obtain:

$$c_{10}(x, y) = \frac{1}{\dot{a}_{00}(x)} \dot{a}_{00}(x) c_{01}(x, y) + \frac{b_{00x}(x, y)}{b_{00y}(x, y)} = 0 \quad (5)$$

$$d_{10}(x, y) = 0 \quad d_{01}(x, y) = \frac{1}{b_{00y}(x, y)}$$

where the dot on top denotes derivation of a one-variable function. Equation (2) now becomes:

$$\begin{aligned} x' &= a_{00}(x) + a_{10}(x, y)p + a_{01}(x, y)q + \dots \\ y' &= b_{00}(x, y) + b_{10}(x, y)p + b_{01}(x, y)q + \dots \\ p' &= \frac{1}{\dot{a}_{00}(x)} p + \frac{b_{00x}(x, y)}{\dot{a}_{00}(x)b_{00y}(x, y)} q + \dots \\ q' &= \frac{1}{b_{00y}(x, y)} q + \dots \end{aligned} \quad (6)$$

Particularly, we conclude that if  $a_{00}$  is a function of  $x$  only,  $a_{00}(x)$ , then  $q'$  must be independent on the first order of  $p$  (since  $d_{10} = 0$ ), and that when  $q = 0$ , the ratio of the direction cosines  $p'$  and  $p$  (which indicates the first-order angular magnification) equals the inverse of the derivative of  $a_{00}(x)$ . The ray-to-ray mapping between  $WF_{in}$  and  $WF_{out}$  is visualized by making  $p = q = 0$  in Eq. (6), so we obtain that also  $p' = q' = 0$  and that:

$$\begin{aligned} x' &= a_{00}(x) \\ y' &= b_{00}(x, y) \end{aligned} \quad (7)$$

If we apply these results to the semi-aplanatic case, in which  $a_{00} = m_x x + l_x$ , we obtain, as expected, that such magnification is the constant  $1/m_x$ , independent of  $x$ . To simplify the nomenclature, in what follows we will call  $a_{00}(x) = a(x)$  and  $b_{00}(x, y) = Y(x, y)$ .

### 3. Particular two-mirror solution

Consider now the particular case where the two coordinate systems have parallel axes as shown in Fig. 1. We are going to design an afocal two-mirror system such that the ray of  $WF_{in}$  that impinge on mirror  $S_1$  at the point  $P$  will be reflected towards point  $Q$  of mirror  $S_2$ , and reflected there again to become a ray of  $WF_{out}$ , such that the  $x$ -coordinate of  $Q$  will only depend on the  $x$ -coordinate of  $P$ . Therefore, in the coordinate system  $x$ - $y$ - $z$  we can write the equations of the two mirror surfaces in parametric form as  $P = (x, y, z(x, y))$  and  $Q = (a(x), Y(x, y), Z(x, y))$ , where  $x$  and  $y$  act as parameters for both surfaces.

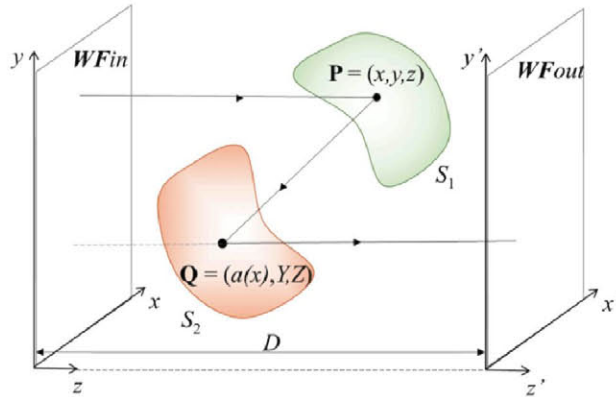


Fig. 1. Two mirror afocal system that transforms  $WF_{in}$  to  $WF_{out}$ .

First, the optical path length between  $WF_{in}$  and  $WF_{out}$  must be constant, and thus:

$$z + L(x, z, Z, Y) + D - Z = C \quad (8)$$

where  $D$  is the constant distance between  $WF_{in}$  and  $WF_{out}$  in Fig. 1 and

$$L(x, z, Z, Y) = \sqrt{(a(x) - x)^2 + (Y - y)^2 + (Z - z)^2} \quad (9)$$

Let us now apply a reflection law on the first surface. It can be computed in the standard vector form or, alternatively, as the derivative of the optical path length with respect to two independent parameters. The latter, with Herzberger notation [10], is written as:

$$\mathbf{r}'_u \cdot \hat{s}' n' - \mathbf{r}_u \cdot \hat{s} n = E_u \quad (10)$$

where vectors  $\hat{s}$  and  $\hat{s}'$  are unit vectors, the primes denoting output space,  $n, n'$  are the refractive indices,  $\mathbf{r}, \mathbf{r}'$  are the position vectors,  $E$  is the optical path length between the position vectors, and  $u$  is the parameter. Thus, for the reflection on the first mirror, taking  $\mathbf{r}' = (x, y, z)$ ,  $\mathbf{r} = (x, y, 0)$ ,  $E = z$  and parameter  $x$  first we have:

$$(1, 0, z_x) \cdot (a(x) - x, Y - y, Z - z) = z_x L(x, z, Z, Y) \quad (11)$$

Eliminating  $L$  from Eq. (8) and Eq. (11), the partial derivative of  $z$  with respect to  $x$  is:

$$z_x = \frac{1}{C - D} (a(x) - x) \quad (12)$$

The analog to Eq. (11) for  $y$  as a parameter is given by:

$$(0, 1, z_y) \cdot (a(x) - x, Y - y, Z - z) = z_y L(x, z, Z, Y) \quad (13)$$

Eliminating  $L$  again with Eq. (8), the partial derivative of  $z$  with respect to  $y$  is:

$$z_y = \frac{1}{C - D} (Y - y) \quad (14)$$

Equations (8), (12) and (14) form the system to be solved with unknown functions  $z, Y$  and  $Z$ . Integrating Eq. (12):

$$z = \frac{1}{C - D} \left( A(x) - \frac{x^2}{2} \right) + z_0(y) \quad (15)$$

where  $A(x) = \int_0^x a(t) dt$  and  $z_0(y)$  is an arbitrary function. Note that  $z(0, y) = z_0(y)$ , so this

function describes the profile of the cross section of mirror  $S_1$  with the plane  $x = 0$ , and can be seen as a boundary condition for the PDE system. Substituting Eq. (15) in Eq. (14):

$$Y = (C - D) \dot{z}_0(y) + y \equiv b(y) \quad (16)$$

So  $Y(x, y)$  is a function of  $y$  only, which we call  $b(y)$ . Finally, from Eq. (8)-(9) we get:

$$Z = z + \frac{1}{2(C - D)} \left( (a(x) - x)^2 + (b(y) - y)^2 - (C - D)^2 \right) \quad (17)$$

and substituting Eq. (16):

$$Z = \frac{A(x)}{C - D} + \frac{a(x)(a(x) - 2x)}{2(C - D)} + z_0(y) + \frac{C - D}{2} (\dot{z}_0^2(y) - 1) \quad (18)$$

Equations (15), (16) and (18) describe the two mirror analytic solution to the problem in which the ray mapping between  $WF_{in}$  and  $WF_{out}$  satisfies  $x' = a(x)$ , independently of  $y$ , with the only condition that the  $x$ - $y$  axes and  $x'$ - $y'$  axes are parallel. We have obtained in the process that this solution also satisfies  $y' = b(y)$ , independently of  $x$  (while in the general case of Eq. (7),  $y'$  may depend on  $x$  too). Therefore this afocal two-mirror system has a decoupled set of ray-to-ray mapping equations of  $x$ -variables and  $y$ -variables, with full symmetry in between them (although our initial condition was asymmetric).

This resulting two-mirror system of Eq. (15), (16) and (18) also has relevant nonimaging properties. When an irradiance distribution is defined on the wavefronts, the energy transfer is controlled by the Jacobian matrix of the mapping function Eq. (1), which for this two mirrors is a diagonal matrix with  $a'(x)$  and  $b'(y)$  as diagonal elements. This nonimaging problem was in fact treated in [11] for irradiance beam shaping when the prescription has rectangular symmetry (although restricted to the case in which the diagonal elements of the Jacobian matrix have equal sign). Notice that the energy transfer problem leads to a second-order Monge-Ampère PDE, while the mapping-constrained problem stated here is of first order only.

For the semi-aplanatic case  $a(x) = m_x x + l_x$ , Eq. (15)-(18) become:

$$z = \frac{1}{C-D} \left( l_x x + (m_x - 1) \frac{x^2}{2} \right) + z_0(y) \quad (19)$$

$$Y = (C-D) \dot{z}_0(y) + y \quad (20)$$

$$Z = \frac{m_x l_x}{C-D} x + \frac{m_x x^2}{2(C-D)} (m_x - 1) + z_0(y) + \frac{l_x^2}{2(C-D)} + \frac{C-D}{2} (\dot{z}_0^2(y) - 1) \quad (21)$$

The parabolic dependence of Eq. (19) with respect to  $x$  implies that in this semi-aplanatic case  $S_1$  is plane-symmetric with respect to the  $x = \text{constant}$  plane passing through the parabola vertex (coinciding with  $x = 0$  when  $l_x = 0$ ), and it is easy to see that  $S_2$  has the same symmetry.

#### 4. Design examples

Figure 2 shows as a first example an afocal freeform semi-aplanatic system that performs the ray-to-ray mapping  $X = 2x/3$  and  $Y = -0.46y^3 + 2.2y - 4.2$ . Integrating Eq. (16) we get:

$$z_0(y) = \frac{1}{C-D} \left( -\frac{0.46y^4}{4} + 0.6y^2 - 4.2y \right) + K \quad (22)$$

Surfaces  $S_1$  and  $S_2$  are given by Eq. (19) and (20). The specific design in Fig. 2 has the parameter  $C-D = 3$  and the arbitrary integration constant  $K = -4$ .

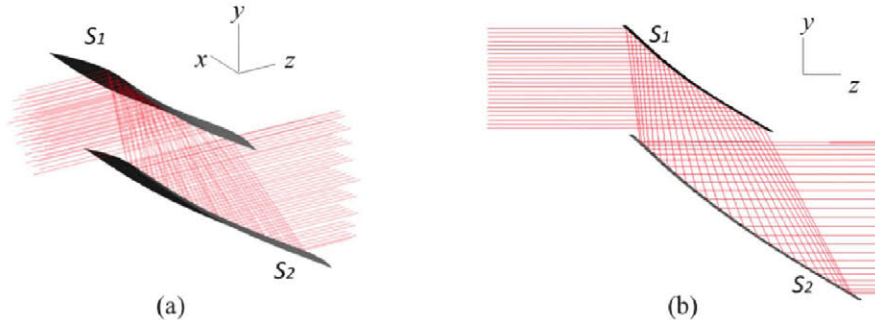


Fig. 2. Semi-aplanatic afocal two freeform mirror system obtained analytically presented in 3D (a) and the 2D cross section with the plane  $x = 0$  (b).



In the previous example, the boundary condition  $z_0(y)$  given in Eq. (21) is a fourth order polynomial, which leads through Eq. (16) to a cubic mapping of the  $y$  coordinates. If we consider next a parabolic boundary condition  $z_0(y)$ , Eq. (16) will imply a linear mapping of the  $y$  coordinates. In Section 3, we found that in this afocal two-mirror  $y' = b(y)$ , independently of  $x$ , so that linear mapping is fulfilled not only along the cross section  $x = 0$ , but for all  $x$  and  $y$  values. Thus, the semi-aplanatic system with parabolic boundary condition is a full aplanatic system. Taking  $y' = m_y y + l_y$  (where  $m_y$  and  $l_y$  are real numbers), then:

$$\dot{z}_0(y) = \frac{1}{C-D}((m_y - 1)y + l_y) \Rightarrow z_0(y) = \frac{1}{C-D} \left( (m_y - 1) \frac{y^2}{2} + l_y y \right) + K \quad (23)$$

When  $m_x = m_y$  it is easy to see that the mirrors are two confocal paraboloids of revolution with axis parallel to the  $z$  axis. However, the magnification  $m_x$  and  $m_y$  can be different, as was already discussed in [8] and in the design of afocal étendue-squeezing microlens arrays in [12], and the resulting mirrors are parabolic-type quadrics without rotational symmetry. From the parabolic dependence of  $z_0(y)$  in Eq. (23) with respect to  $y$  it is easy to deduce that in this aplanatic case  $S_1$  and  $S_2$  are not only plane-symmetric with respect to a plane  $x = \text{constant}$  but also with respect to a plane  $y = \text{constant}$  (coinciding with  $y = 0$  when  $l_y = 0$ ). Plane-symmetric two-mirror systems were already studied in [13], restricted to off-axis rotationally symmetric conics with sharp imagery through second order in both spatial and angular coordinates (while the spatial coordinates are not limited is the present work).

Figure 3 shows a specific example of such an afocal aplanatic two mirror design for parameters  $C - D = 10$ ,  $K = 3$ ,  $l_x = 0$ ,  $l_y = 0$  and magnifications  $m_x = -2$  and  $m_y = 2/3$ . The mirrors are hyperbolic paraboloids with axes parallel to the  $z$  axis.

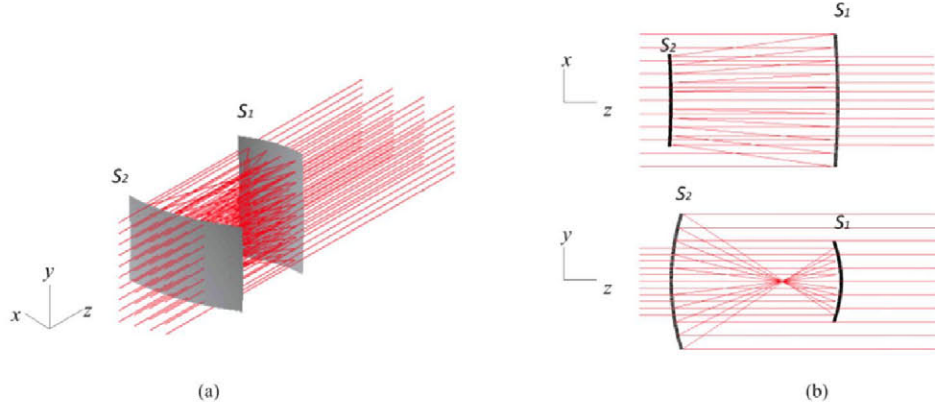


Fig. 3. Analytically obtained two-freeform-mirror aplanatic system in 3D (a); and two 2D cross sections with planes  $y = 0$  and  $x = 0$  (b).

## 5. Discussion

The second selected design shown in Section 4 (Fig. 3) is, to our knowledge, the first example of an aplanatic system made of only two freeform (non-rotationally symmetric) mirrors. This seems to contradict that in general at least three freeform optical surfaces are necessary to design an aplanatic system [8]. However, the necessity of three freeform surfaces for the general case does not impede that there are particular cases in which fewer surfaces can provide an aplanatic design. In fact, apart from the trivial example of the flat mirror, two other particular cases are already known: the two-mirror rotational case mentioned in the introduction and the aplanatic points of the refractive sphere (using a single surface) [1].

Although in general forcing that an afocal system has a mapping such that  $x'$  is a function of  $x$  only does not impose any condition on  $y'$ , when the  $x$ - $y$  axes are parallel to the  $x'$ - $y'$  axes

and two mirrors are used,  $y'$  results to be a function of  $y$  only. As a consequence, the two-mirror semi-aplanatic system was automatically aplanatic. If there are other semi-aplanatic configurations in which this also occurs it is an open problem that deserves further research.

## **6. Conclusions**

An analytic solution of first-order partial differential equations for an afocal two freeform mirror design problem was presented. If the ray-to-ray mapping between a selected input and output stigmatic wavefront pair is such that the output  $x$ -coordinate of a ray is a function of only the  $x$ -coordinate of its corresponding input ray (independently of its  $y$ -coordinate), the  $y$ -coordinate of the output ray also results in being a function of only the  $y$ -coordinate of the output ray in these two-mirror systems. Furthermore, for a semi-aplanatic case, choosing a specific boundary condition, a full freeform two-mirror aplanat is achieved, which is to our knowledge the first reported full aplanatic two-mirror system without rotational symmetry. The systems found in this paper have multiple applications; as beam shaping [11], étendue squeezing [12] and anamorphic imaging systems.

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